Quantum mechanics II, Problems 12 : Fermions and Bosons

TA : Slimane Thabet, Sofia Brizigotti, Alba Miren Taddei, Reyhaneh Aghaei Saem, Mehrad Sahebi, Ricard Puig, Sacha Lerch

Problem 1 : Warm up

Boson. Imagine we have two photons in two spatial modes labelled $|\mathbf{k}_1\rangle$ or $|\mathbf{k}_2\rangle$ with horizontal and/or vertical polarizations. Which of these is a valid symmetric state for the two photons ?

(1) $|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|\mathbf{k}_1, \mathbf{k}_2\rangle + |\mathbf{k}_2, \mathbf{k}_1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|H, V\rangle + |V, H\rangle\right)$
(2) $|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|\mathbf{k}_1, \mathbf{k}_2\rangle - |\mathbf{k}_2, \mathbf{k}_1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|H, V\rangle - |V, H\rangle\right)$
(3) $|\psi_3\rangle = |\mathbf{k}_1, \mathbf{k}_2\rangle \otimes \frac{1}{\sqrt{2}}\left(|H, V\rangle + |V, H\rangle\right)$
(4) $|\psi_4\rangle = \frac{1}{\sqrt{2}}\left(|\mathbf{k}_1, \mathbf{k}_2\rangle + |\mathbf{k}_2, \mathbf{k}_1\rangle\right) \otimes |H, V\rangle$

Fermions. Consider two electrons in Helium which are in two different orbital states (one in the ground state, one in an excited state), such as we labeled $|a\rangle$, $|b\rangle$ above with up and/or down spins. Which 2 of the following 6 states are possible states of the two electrons ? Justify.

(1) $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|a, b\rangle + |b, a\rangle) \otimes \frac{1}{\sqrt{2}}\left(|\uparrow, \uparrow\rangle + |\downarrow, \downarrow\rangle\right)$
(2) $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|a, b\rangle + |b, a\rangle) \otimes \frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle)$
(3) $|\psi_3\rangle = \frac{1}{\sqrt{2}}(|a, b\rangle - |b, a\rangle) \otimes \frac{1}{\sqrt{2}}\left(|\uparrow, \uparrow\rangle + |\downarrow, \downarrow\rangle\right)$
(4) $|\psi_4\rangle = \frac{1}{\sqrt{2}}(|a, b\rangle - |b, a\rangle) \otimes \frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle)$
(5) $|\psi_5\rangle = \frac{1}{\sqrt{2}}(|a, b\rangle - |b, a\rangle) \otimes \frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle)$
(6) $|\psi_6\rangle = |a, b\rangle \otimes \frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle)$

Problem 2 : Bosonic wavefunction

Consider the possible basis states for a system of n Bosons. It can be written as

$$|\psi_{\mathbf{x}}\rangle = \mathcal{N} \sum_{\mathbb{P} \in S_n} \mathbb{P}|x_1, x_2, \ldots, , x_n\rangle = \mathcal{N} \sum_{\mathbb{P} \in S_n} |x_{\mathbb{P}(1)}\rangle|x_{\mathbb{P}(2)}\rangle \ldots |x_{\mathbb{P}(n)}\rangle, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathcal{N}$ is a normalization factor, and $\mathcal{S}_n$ is the symmetric group on n elements.

Prove that the normalization factor is

$$\mathcal{N} = \frac{1}{\sqrt{n!}\sqrt{\prod_k n_k!}} \tag{2}$$

where n_k is repeated entries and $\sum_k n_k = n$.

Problem 3 : Eigenspectra of Fermions and Bosons

We are familiar with simple harmonic oscillators from Quantum Physics 1. A particle with mass m in one-dimensional harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$ has the following eigenenergies.

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, \ldots \tag{3}$$

and we call $\varphi_n(x)$ the associated eigenfunctions which we suppose normalized.

Now consider that we want to study a system of identical particles that do not interact so that we can consider the total Hamiltonian of these systems as a sum of one particle Hamiltonian.

Bosons

We take the particles as Bosons of zero spin here. So the eigenfunction of Bosons needs the spatial part which must be symmetrical and we know the one particle case of them from $\varphi_n(x)$.

1. Consider the case of 2 Bosons. Find the first 3 energy levels, their degeneracy, and the associated eigenfunctions.

   (Hint : Use $\varphi_n(x)$s to make two-particle symmetric wavefunctions.)

2. We are now interested in the case of 3 Bosons. Find the first 3 energy levels, their degeneracy, and the associated eigenfunctions.

Fermions

In this section, we are interested in the case of two electrons, i.e. two Fermions of spin of spin 1/2, and we are looking for the eigenfunctions of two particles in the form

$$\psi(x_1, x_2, s_1, s_2) = \varphi(x_1, x_2)\chi(s_1, s_2), \tag{4}$$

where x_i denote the position of the particles and $s_i \in \{+\frac{1}{2}, -\frac{1}{2}\}$ their spins. $\varphi(x_1, x_2)$ is the spatial part and $\chi(s_1, s_2)$ is the spin part of the wavefunction.

1. $\chi(s_1, s_2)$ means that the first particle has spin s_1 and the second particle has spin s_2. What is the dimension of the space of functions $\chi(s_1, s_2)$?

2. We define in this space, the operators $\mathbb{P}$ (permutation) and S^z (total spin in the $\hat{z}$ direction) by :

   $$\begin{aligned} \mathbb{P}\chi(s_1, s_2) &= \chi(s_2, s_1) \\ S^z\chi(s_1, s_2) &= (s_1 + s_2)\chi(s_1, s_2) \end{aligned} \tag{5}$$

   In other words, we know that $\chi(s_1, s_2)$ is the eigenfunction of S^z with eigenvalue $s_1 + s_2$.

   (a) Demonstrate that we can diagonalize $\mathbb{P}$ and S^z in a common basis.

   (b) Using the previous result find the common eigenstates of $\mathbb{P}$ and S^z.

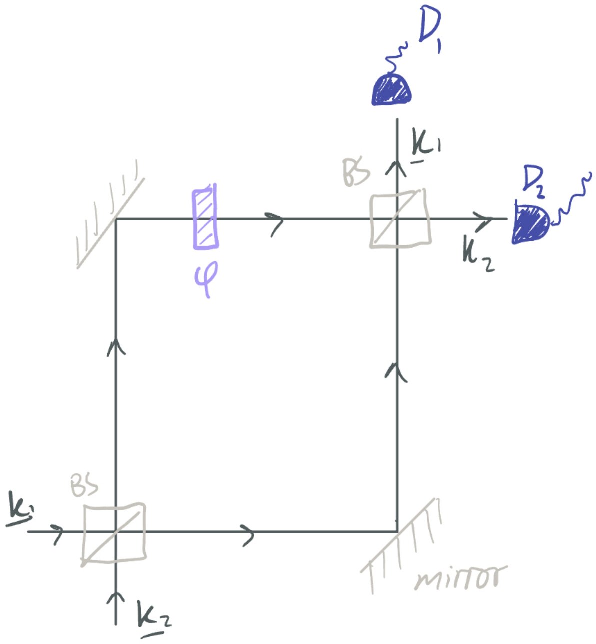
   (Hint : S^z is already diagonal in the basis of $\chi(s_1, s_2)$ so try to diagonalize $\mathbb{P}$ in this basis.)

3. We now come back to the two electrons in a harmonic potential problem. Find the first 3 energy levels, their degeneracy, and the associated eigenfunctions.

   (Hint : Find antisymmetric wavefunctions by trying the possible combination of spatial and spin parts.)

Problem 4 : Mach-Zehnder interferometer

The Mach-Zehnder interferometer consists of two 50 :50 beamsplitters with a phase shifter in one arm, arranged as follows :



When working in the second quantization it is often helpful to work in the Heisenberg picture and consider the action of any unitary process on the creation and annihilation operators rather than on a given state directly. The phase shift of φ in the upper arm in the Heisenberg picture acts as $\hat{a}^\dagger_{k_2} \to e^{i\varphi}\hat{a}^\dagger_{k_2}$. It follows that the total unitary matrix on the mode operators describing the apparatus is

$$U = U_{BS2}\,U_{phaseshift}\,U_{BS1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{6}$$

Where $U_{BS2} = U^\dagger_{BS1}$ and U_{BS1} is the effect of beamsplitter on the mode operators $(\hat{a}^\dagger_{k_1}, \hat{a}^\dagger_{k_2})$. In other words, we know that in the Heisenberg picture, the creation operators are changed as follows

$$\hat{a}^\dagger_{k_1} \xrightarrow{BS1} \frac{1}{\sqrt{2}}(\hat{a}^\dagger_{k_1} + \hat{a}^\dagger_{k_2}), \tag{7}$$

$$\hat{a}^\dagger_{k_2} \xrightarrow{BS1} \frac{1}{\sqrt{2}}(\hat{a}^\dagger_{k_1} - \hat{a}^\dagger_{k_2}). \tag{8}$$

So we can write it in the following way

$$\begin{pmatrix} \hat{a}^\dagger_{k_1} \\ \hat{a}^\dagger_{k_2} \end{pmatrix} \xrightarrow{BS1} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \hat{a}^\dagger_{k_1} \\ \hat{a}^\dagger_{k_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\hat{a}^\dagger_{k_1} + \hat{a}^\dagger_{k_2}) \\ \frac{1}{\sqrt{2}}(\hat{a}^\dagger_{k_1} - \hat{a}^\dagger_{k_2}) \end{pmatrix}. \tag{9}$$

We can also see the same calculations to find the unitary matrix of the phase shift.

$$\hat{a}^\dagger_{k_1} \xrightarrow{phase} \hat{a}^\dagger_{k_1} \tag{10}$$

$$\hat{a}^\dagger_{k_2} \xrightarrow{phase} e^{i\varphi}\hat{a}^\dagger_{k_2}. \tag{11}$$

So we can write it in the following way

$$\begin{pmatrix} \hat{a}^\dagger_{k_1} \\ \hat{a}^\dagger_{k_2} \end{pmatrix} \xrightarrow{phase} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \begin{pmatrix} \hat{a}^\dagger_{k_1} \\ \hat{a}^\dagger_{k_2} \end{pmatrix} = \begin{pmatrix} \hat{a}^\dagger_{k_1} \\ e^{i\varphi}\hat{a}^\dagger_{k_2} \end{pmatrix}. \tag{12}$$

(i) Suppose one photon is input into the mode $\mathbf{k}_1$. What is the probability as a function of ϕ of finding it at each of the detectors D_1 and D_2 ?

(ii) Suppose one photon is input into mode $\mathbf{k}_1$ and one photon is input into mode $\mathbf{k}_2$. What are the probabilities of finding 2, 1, or 0 photons at each of the detectors D_1 and D_2 ?

(iii) For each of the cases in (i) and (ii), imagine that just after the phase shifter a detector D_3 was inserted. Assuming it does not detect any photons, how are the probabilities of finding the photons at detectors D_1 and D_2 affected ?

(iv) What is the relationship between this setup and the two-slit experiment ?